Today: §2.6: The Squeeze Theorem and Trig Limits
§2.8: Intermediate Value Theorem

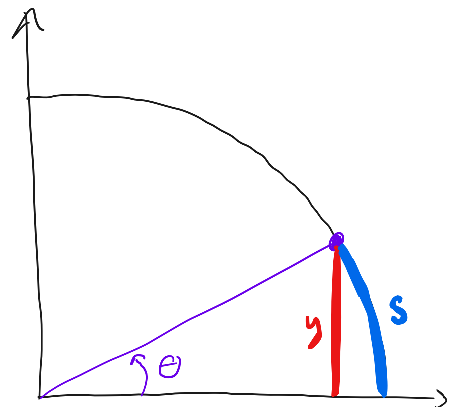
Big goal for today: show that

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0.$$

⚠ Warning: tricky trig ahead!

Zoom poll – try it if you have time!

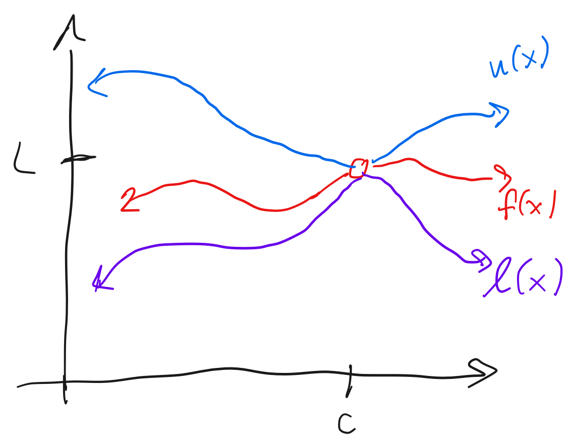
Warm-up: For $0 < \theta < \frac{\pi}{2}$, which is larger, $\underbrace{\theta}_{\text{Option 1}}$ or $\underbrace{\sin(\theta)}_{\text{Option 2}}$?



Hint: What is the length **y** of the straight line?

What is the length **s** of the curved line?

Squeeze Theorem If $\ell(x) \le f(x) \le u(x)$ for all $x \neq c$ in an interval containing c, and $\lim_{x\to c} \ell(x) = \lim_{x \to c} u(x) = L$, then $\lim_{x \to c} f(x) = L$.



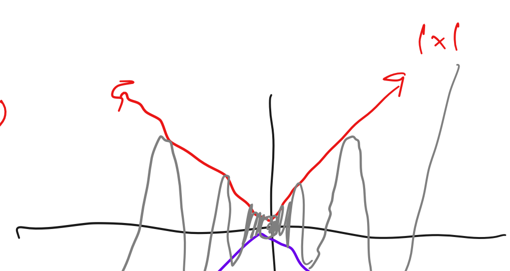
Ex (1) $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$

Product Law? $\left(\lim_{x\to 0} x\right)\underbrace{\left(\lim_{x \to 0} \sin\left(\frac{1}{x}\right)\right)}_{\text{DNE}}$

$-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all $x \neq 0$

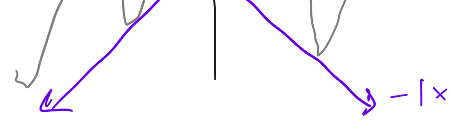
$\Rightarrow -|x| \le x \cdot \sin\left(\frac{1}{x}\right) \le |x|$ " " $x \neq 0$

↑ be careful with $x < 0$



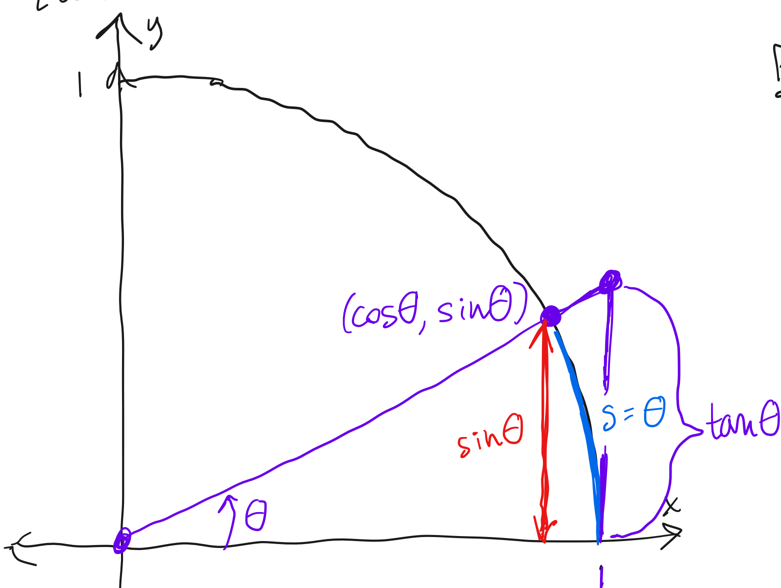
Check $\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x|$

by Squeeze $\Longrightarrow \lim_{x\to 0} x\cdot\sin\left(\frac{1}{x}\right) = \boxed{0}$



Thm $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1.$

Proof idea: "sandwich" or "squeeze" $\frac{\sin\theta}{\theta}$ between two functions that are simpler as $\theta \to 0$.



$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\text{opp}}{1} = \text{opp}$

$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$

Fact about radians: $\theta = \frac{s}{\cancel{r}\,1} = s$

For $0 \le \theta \le \pi/2$:

- $\theta \ge \sin\theta \Rightarrow \boxed{\underbrace{\frac{\sin\theta}{\theta}}_{} \le \underbrace{1}_{u(\theta)}}$

  (Halfway there! $1 \ge \frac{\sin\theta}{\theta}$

- visually: vertical line $\tan\theta$ longer than $s=\theta$ – done more carefully in book.

  $\Longrightarrow s = \theta \le \tan\theta = \frac{\sin\theta}{\cos\theta}$

  $\Longrightarrow \theta\cos\theta \le \sin\theta$

  $\Longrightarrow \boxed{\underbrace{\cos\theta}_{\ell(\theta)} \le \frac{\sin\theta}{\theta}}$

Notice $\lim_{\theta\to 0^+} \underbrace{u(\theta)}_{1} = 1$ and $\lim_{\theta\to 0^+} \underbrace{\ell(\theta)}_{\cos\theta} = \cos(0) = 1$

Thus $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$

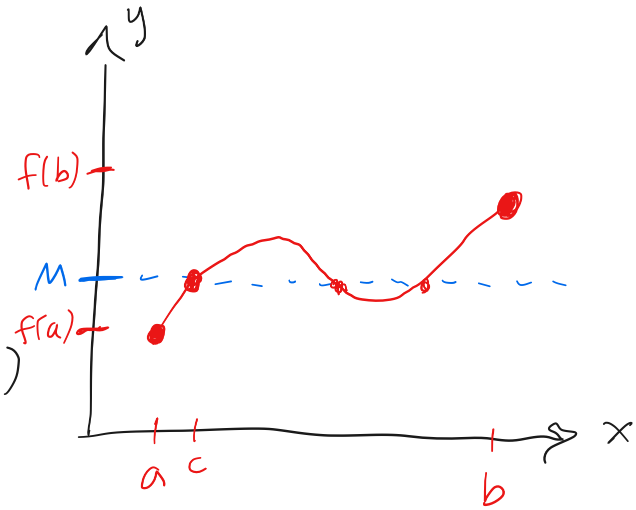
(Also works for $\theta < 0$)

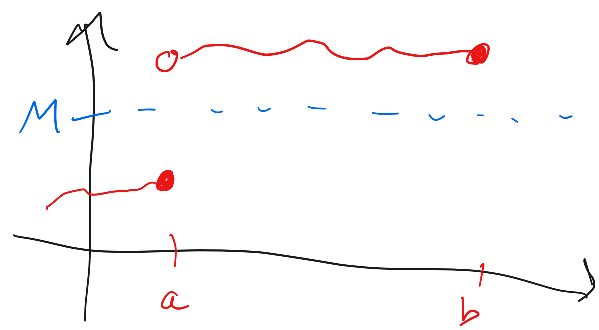
Theorem $\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = 0$ $\quad \xrightarrow{\text{Indeterminate}} \frac{1-\cos(0)}{0} = \frac{0}{0}$

Proof algebraic manipulation plus $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

§2.8 Intermediate Value Theorem (IVT)

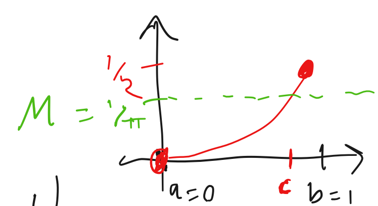
Given a function f continuous on $[a,b]$ with $f(a) \neq f(b)$, if M is a number between $f(a)$ and $f(b)$, then there is <u>at least one</u> point $c \in (a,b)$ at which $f(c) = M$.





<u>Ex</u> Show that $f(x) = \frac{x^2}{x^5+1}$ is equal to $\frac{1}{\pi}$ for some $x \in [0,1)$.

- $f(x)$ continuous on $[0,1]$ ✓
  True for any rational function as long as denom. is nonzero
- $f(0) = \frac{0}{0+1} = 0$ and $f(1) = \frac{1}{1+1} = \frac{1}{2}$
- $M = \frac{1}{\pi} \approx \frac{1}{3.14} < \frac{1}{2}$. <u>Notice</u> $\frac{1}{\pi} \in (0,1)$.



<u>So</u> by the IVT, $f(x)$ must be equal to $\frac{1}{\pi}$ at some value c in $(0,1)$.

<u>Ex</u> Show that $p(x) = x^5 - 3x^2 - x + 2$ <u>has a zero</u>.

$p(0) = 2$

$p(-1) = -1$